

# On the multipole moments of a rigidly rotating fluid body

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## Abstract

Based on numerical simulations and analytical calculations we formulate a new conjecture concerning the multipole moments of a rigidly rotating fluid body in equilibrium. The conjecture implies that the exterior region of such a fluid is not described by the Kerr metric.

## 1 Introduction

Bardeen and Wagoner [1] observed in numerical tests in 1971 that the quadrupole moment of a rigidly rotating disk of dust is always greater than that of the Kerr metric with the same angular momentum and mass except in the extreme relativistic case, when they become the same,

$$|Q_2^{\text{B\&W}}| \geq |Q_{2,\text{Kerr}}^{\text{B\&W}}|. \quad (1)$$

As already concluded by Bardeen and Wagoner, the general expectation is that this also holds for other spacetimes containing a central rigidly rotating perfect fluid. The recent work of Bradley and Fodor [2] lends support for this in the slowly rotating case.

It is now interesting to ask the question whether this holds not only for the quadrupole moment but for every moment,

$$|Q_n| \geq |Q_{n,\text{Kerr}}|. \quad (2)$$

In this paper, based upon [3], we will formulate this conjecture and collect evidence in favor of it from numerical and analytical results. We will make use of geometrical units with  $G = c = 1$ .

## 2 Formulation of the conjecture

The multipole moments calculated by Fodor et al. in [4] are equivalent to the invariantly defined ones by Geroch and Hansen [5, 6] for axially symmetric and stationary spacetimes. Because of the form of the Ernst potential on the axis, the mass- and rotation moments are in this context given by

$$Q_{n,\text{Kerr}} = i^n \frac{J^n}{M^{n-1}} \equiv M_n + iJ_n. \quad (3)$$

In accordance with relation (2) we state:

**Generalized Quadrupole-Conjecture.** *For axially symmetric, stationary and asymptotically flat spacetimes with angular momentum  $J$ , mass  $M$  and multipole moment  $Q_n$*

$$A_n := \left| \frac{J^n}{M^{n-1} \cdot Q_n} \right| \leq 1 \quad (4)$$

*always holds if the spacetime is that of a rigidly rotating perfect fluid body in equilibrium, surrounded by vacuum.*

Furthermore, in accordance with our experience, the equality is only reached in the case of a black hole limit, which is then necessarily an extreme Kerr black hole, see [7]. In general, the exterior spacetime differs from the Kerr metric.

## 3 Evidence

### 3.1 Newtonian Limit

In the Newtonian limit, a rigidly rotating object has angular momentum

$$J = \Theta \cdot \Omega \quad (5)$$

with moment of inertia  $\Theta$  and angular velocity  $\Omega$ . In this limit we can compute a kinetic energy and define a characteristic velocity via

$$2E_{\text{kin}} = \Theta \cdot \Omega^2 =: M \cdot v_{\text{char}}^2. \quad (6)$$

Restricting ourselves to axial and equatorial symmetry, relation (4) becomes for  $n = 2, 4, \dots$

$$\left| \frac{J^n}{M^{n-1} \cdot Q_n} \right| = \left| \frac{\Theta^n \Omega^n}{M^{n-1} \cdot Q_n} \right| = \left| \frac{\Theta^{n/2}}{M^{n/2-1} \cdot Q_n} \right| v_{\text{char}}^n \leq 1, \quad (7)$$

which should always hold under the condition of small velocities  $v_{\text{char}} \ll 1$  provided  $\Theta^{n/2} M^{1-n/2} Q_n^{-1}$  is limited.

### 3.2 Numerical tests

With help of the numerical program described in [8, 9] we were able to test the conjecture for different equations of state and different topologies in the case of the quadrupole and octupole moments. Exemplarily we will show some of the results to underline the conjecture.

The tests covered three equations of state for spheroidal stars containing homogeneous matter, an MIT-Bag model equation of state for quark matter following [10] and a model for a completely degenerated ideal neutron gas following [11] and additionally a quadrupole-test for thin rotating rings.

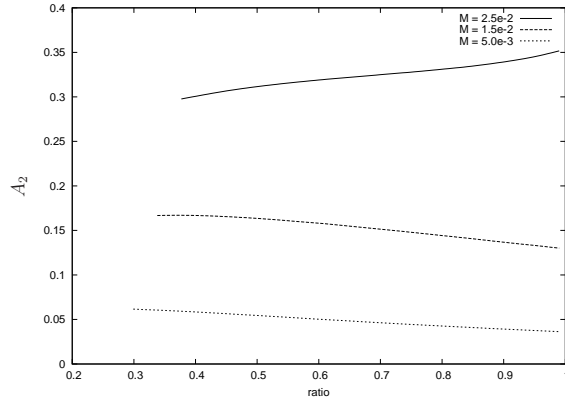


Figure 1:  $A_2$  for quadrupole moments of strange stars with an MIT-Bag model equation of state and varying masses depending on the ratio of polar to equatorial radius.

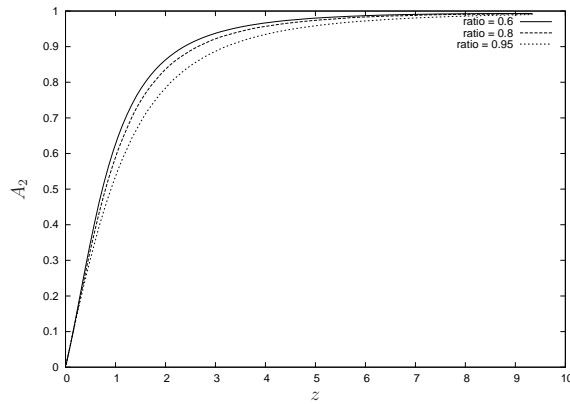


Figure 2: Rotating homogeneous rings for different ratios of radii depending on the redshift parameter  $z$ , see [3].

Figures 1 and 2 show  $A_2$  for the cases of a star with an MIT-Bag model equation of state and homogeneous rings, respectively.

Figure 3 shows a test in the case of the octupole moment for homogeneous stars with different masses.

### 3.3 The rigidly rotating disk of dust

The rigidly rotating disk of dust was solved analytically by Neugebauer and Meinel [12, 13]. The associated multipole moments were derived shortly thereafter by Klei-wächter et al. [14], see also [15]. Figure 4 shows that in this case the conjecture is true for  $n = 2 \dots 10$ . Moreover, it is interesting that we have  $A_n \geq A_{n+1}$ .

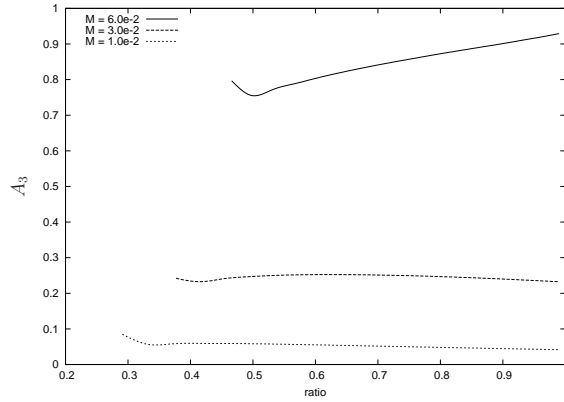


Figure 3:  $A_3$  for homogeneous stars with different masses.

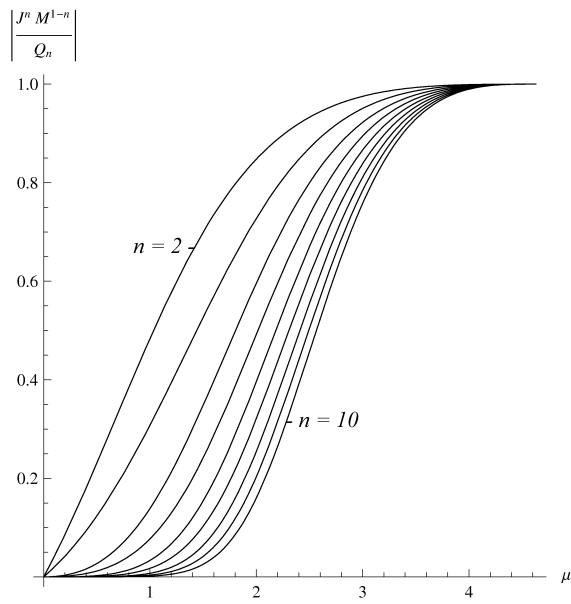


Figure 4: The rigidly rotating disk of dust: The conjecture has been verified up to  $n = 10$ . The solution depends on a parameter  $\mu$  ( $0 < \mu < \mu_0 = 4.62966 \dots$ ), where  $\mu \ll 1$  corresponds to the Newtonian limit and  $\mu \rightarrow \mu_0$  leads to the black hole limit, see [12], [16].

## 4 Conclusions and remarks

Since it is easy to construct solutions of the vacuum Einstein-equations violating the conjecture, e.g. with the algorithm presented by Manko and Ruiz in [17], it will be interesting to investigate, which requirements on the sources are necessary to ensure  $A_n \leq 1$  and which are not necessary.

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